RBE 2004

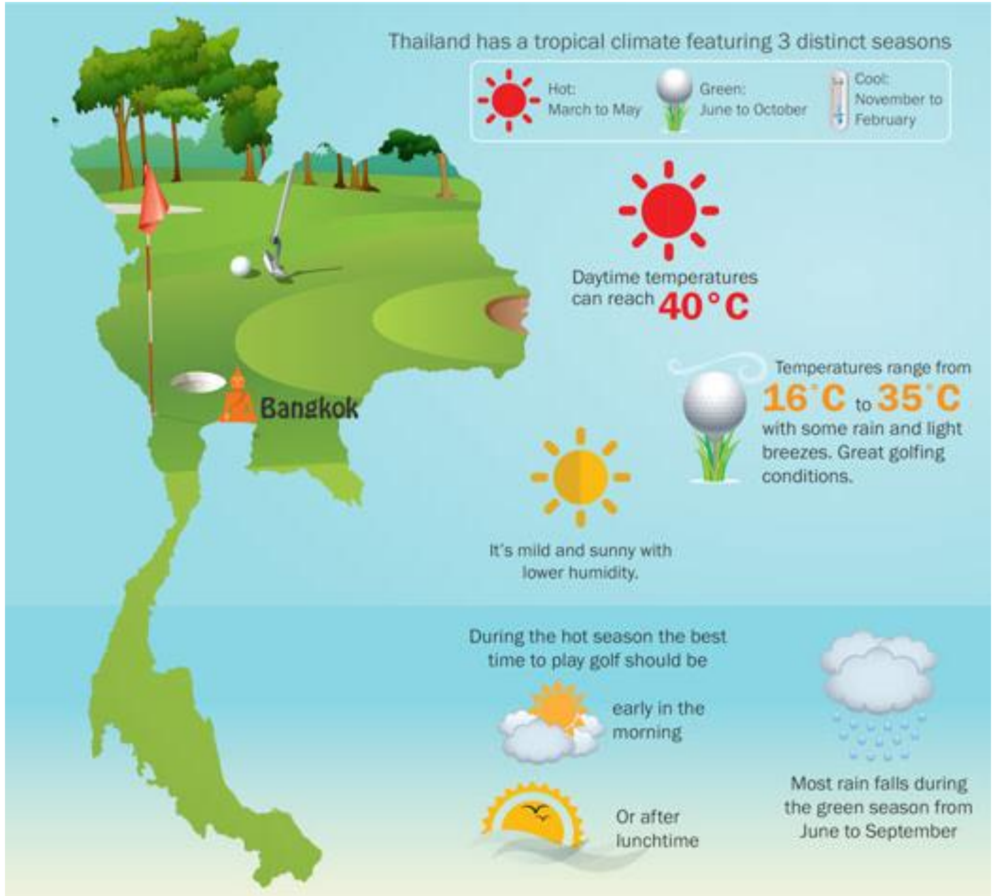
ระบบอัตโนมัติ (Automatic System)

สาขาวิศวกรรมหุ่นยนต์
คณะวิศวกรรมศาสตร์และเทคโนโลยีอุตสาหกรรม
มหาวิทยาลัยราชภัฏสวนสุนันทา

Chapter 2 Mathematical Model of Systems Lecture 2

วัตถุประสงค์

- นิยามศัพท์ โมเดล (“Model”) และการใช้โมเดลในการวิเคราะห์ปัญหาของระบบพลศาสตร์ที่สนใจ
- การสร้างโมเดลในรูปแบบสมการอนุพันธ์ (differential equations)
- การใช้ Laplace Transform
- การทดสอบและวิเคราะห์ระบบโดยใช้อินพุทชนิดต่างๆ



- Question 1: how much will it rain tomorrow?
- Question 2: will it rain in the next 5-10 days?
- Question 3: will we have a drought next summer?

Different questions ⇒ different models

Intro lecture

- Quest to find a Math eq to explain a physical system
- Differential equation usually suitable
- Relationship between variables those describe the system
- Necessitate assumption to be made to overcome the complexity of the system
- Linearize the system then Laplace transform can be used to find the response of the system

ดังนั้น

โมเดลคือสมการคณิตศาสตร์ที่ใช้อธิบายลักษณะ พฤติกรรม ความเป็นไปของระบบพลศาสตร์ (ในวิชานี้จะจำกัดแค่ระบบทางวิศวกรรม)

โดยอาศัยทฤษฎีทางฟิสิกส์ซึ่งศึกษาระบบนั้นๆเช่น ทางกล ไฟฟ้า แม่เหล็ก เป็นต้น เราจะได้ความสัมพันธ์ของตัวแปรและพารามิเตอร์ซึ่งอธิบายพฤติกรรมของระบบ ธรรมชาติของระบบ ความสัมพันธ์นี้จะเขียนออกมาเป็นสมการซึ่งโดยปกติสมการนี้จะเป็นสมการอนุพันธ์ เพราะระบบพลศาสตร์มีพฤติกรรมที่แปรเปลี่ยนไปตามเวลา โมเดลทำให้เราเข้าใจความเป็นไปของปรากฏการณ์ทางธรรมชาติต่างๆ และสามารถคาดเดาสิ่งที่อาจจะเกิดขึ้นในอนาคตได้อีกด้วย   กล่าวคือโมเดลก็คือสมการคณิตศาสตร์ สามารถใช้แทนกันได้

ในระบบควบคุม

โมเดล = สมการคณิตศาสตร์ = plant = process

Modeling of Physical Systems

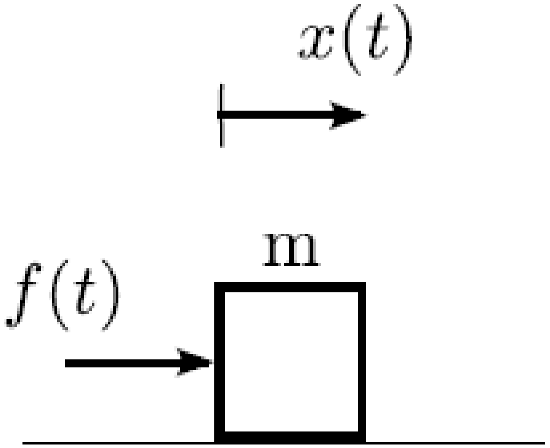
- Differential Equations
  - Mechanical System : translation, rotation by Newton Laws
  - Electrical System : RLC circuit by Kirchoff's Law
  - Fluid System : fluid flow system by Bernoulli's principle
  - Heat System : heating system by Fourier Laws

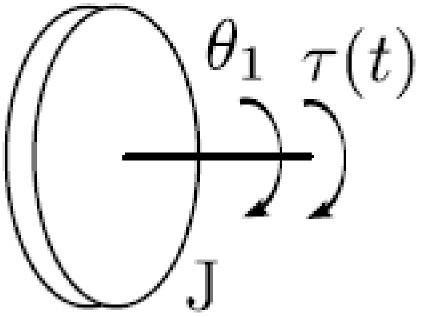
Elements of a mechanical system

→ **Mass**: The quantity of matter in a body

→ **Inertia**: Tendency to resist changes in state of motion

Idealization: Rigid body



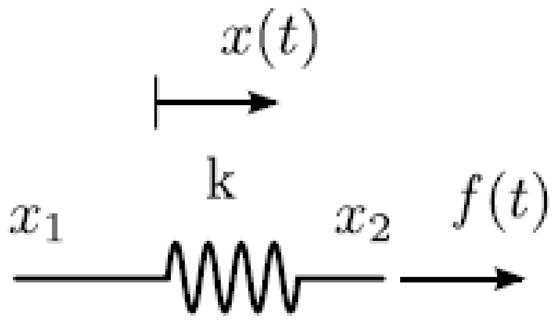


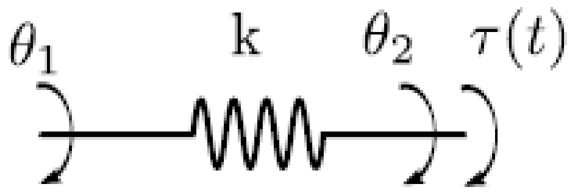
Find the differential equation that describes the behavior of a physical system

Elements of a mechanical system

→ **Spring**: Designed to store energy

Idealization: Negligible mass and damping

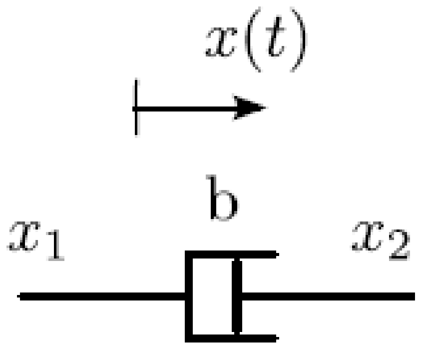


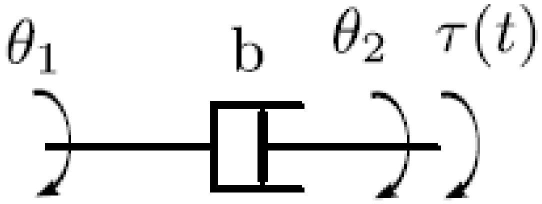


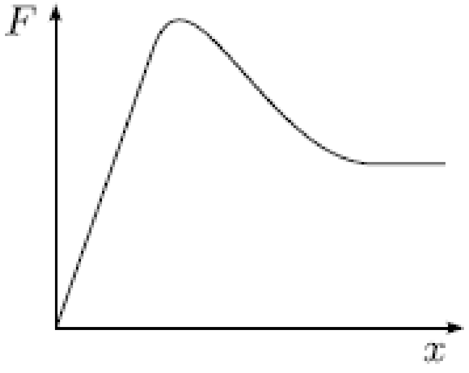
Elements of a mechanical system

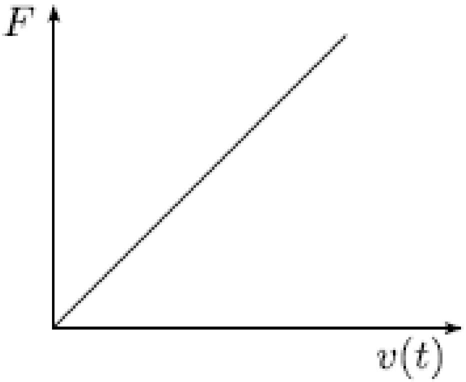
→ **Viscous damper**: Designed to dissipate energy

Idealization: negligible mass and stiffness





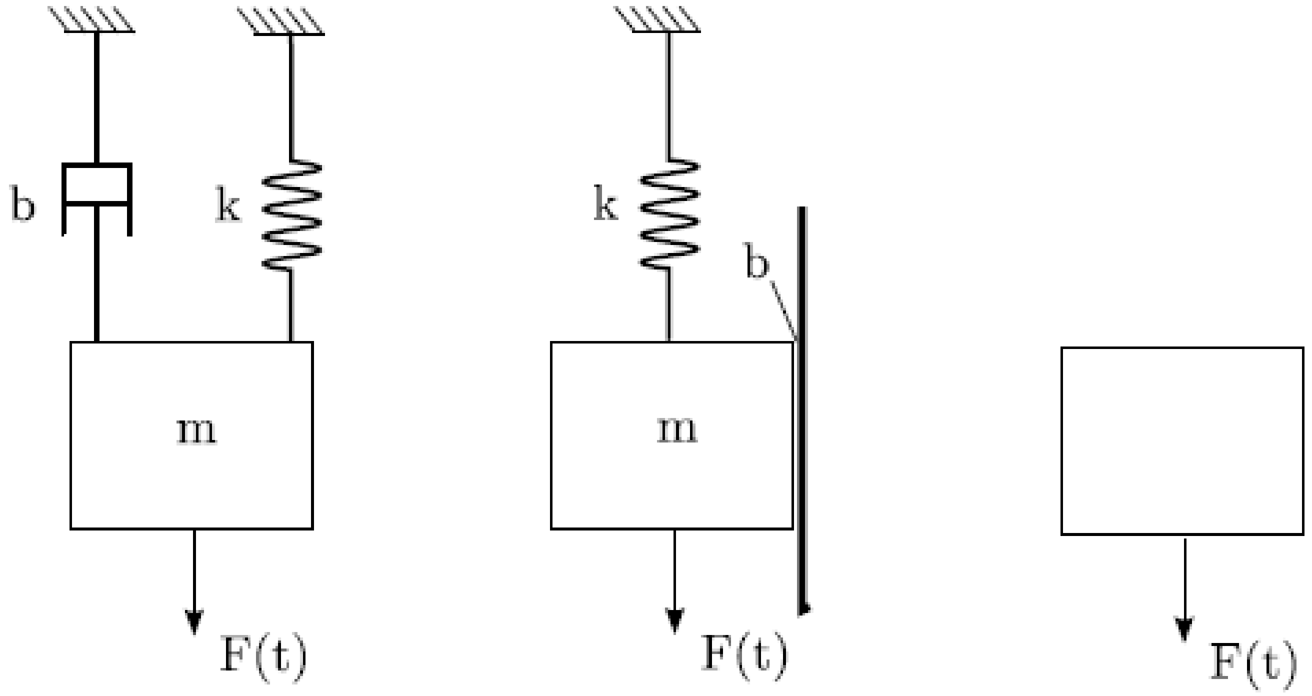




viscous friction
$\neq$
kinetic/static friction

Example

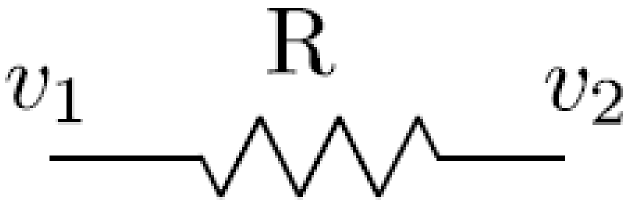
Find the equation of motion of the spring-mass-damper system.



Elements of electrical circuits

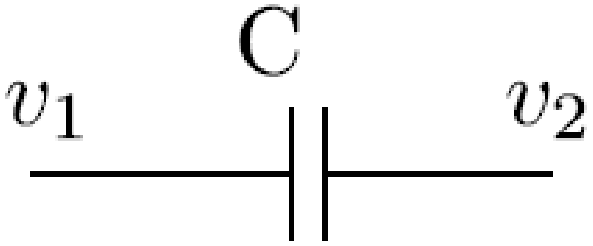
→ **Resistor**: Resistance against electric current

Idealization: No inductance or capacitance



→ **Capacitor**: Stores energy in an electric field

Idealization: No inductance or resistance

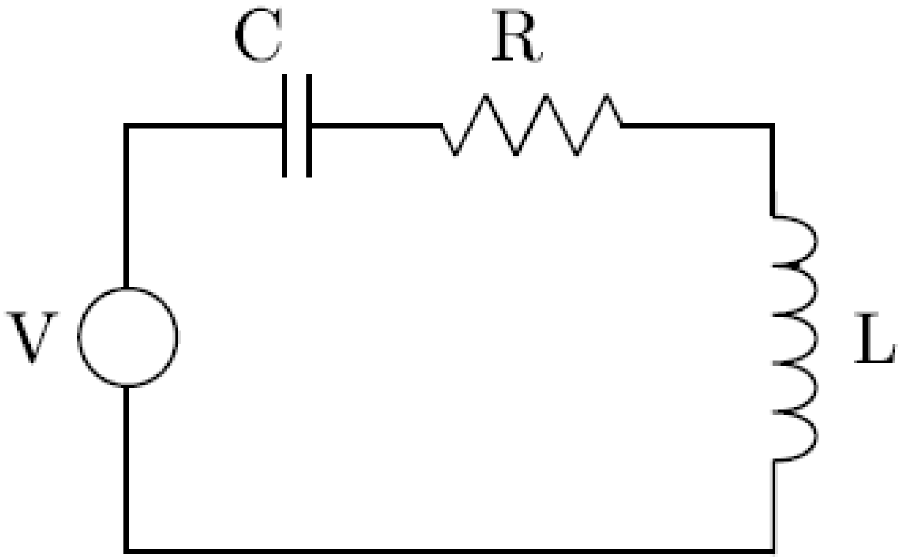


→ **Inductor**: Stores energy in a magnetic field

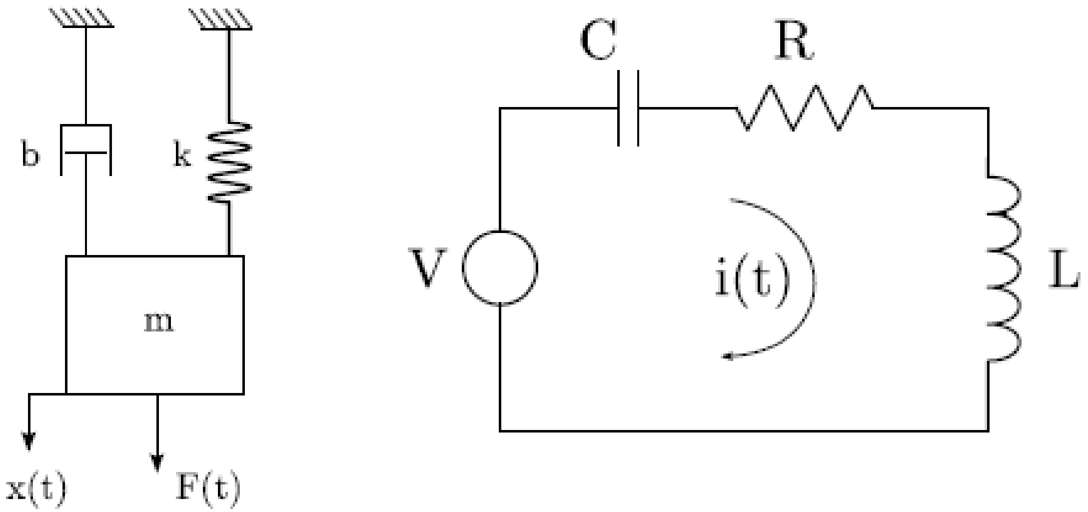
Idealization: No capacitance or resistance

Example

Find the relation between the voltage V, the current, and the charge in the circuit.



Mechanical/electrical analogy



$$F = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx \tag{1}$$

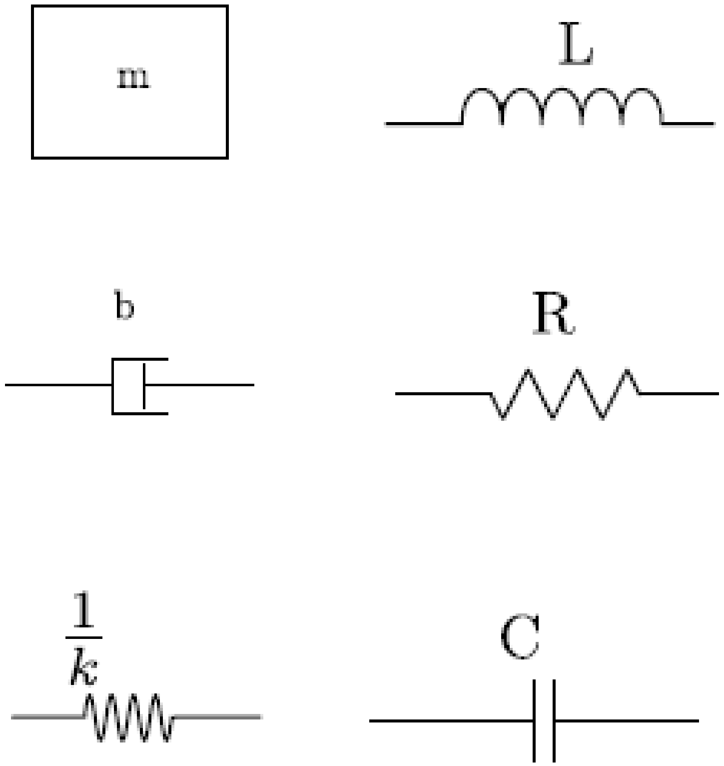
$$V = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q \tag{2}$$

Impulse response

$$x(t) = Ke^{-\alpha t}\sin(\beta t + \theta) \tag{3}$$

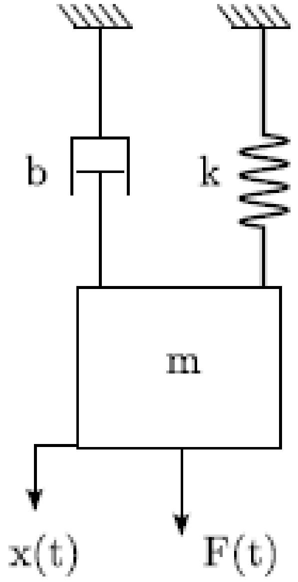
Mechanical/electrical analogy

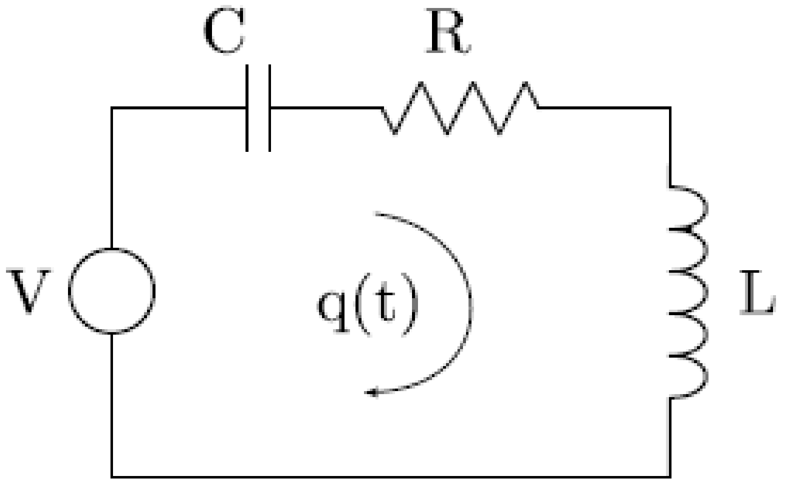
Mechanical		Electrical	
Force	F	Voltage	V
Velocity	v	Current	i
Displacement	x	Charge	q
Damping	b	Resistance	R
Mass	m	Inductance	L
Compliance	k^{-1}	Capacitance	C

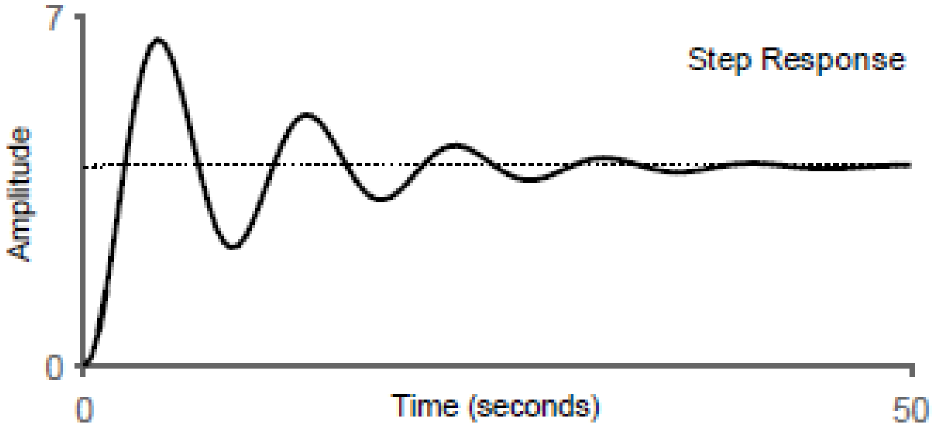


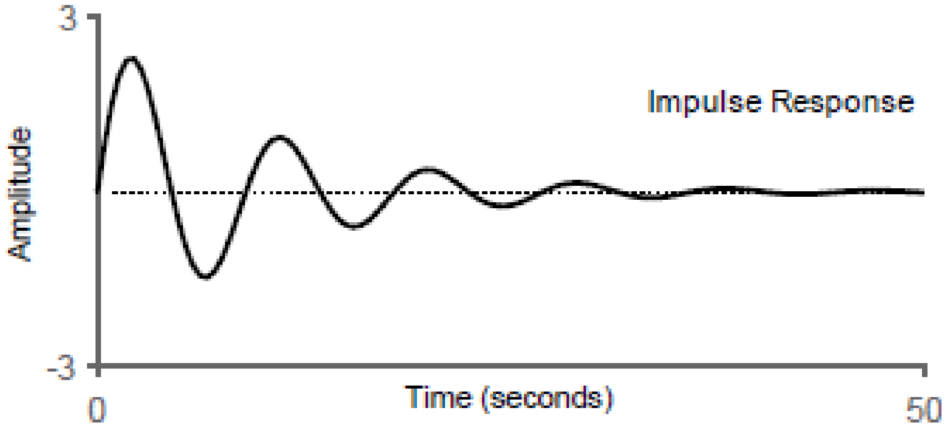
Mechanical/electrical analogy

Taking: $m = 5$ kg, $k = 0.25$ N/m, $b = 0.1$ Ns.



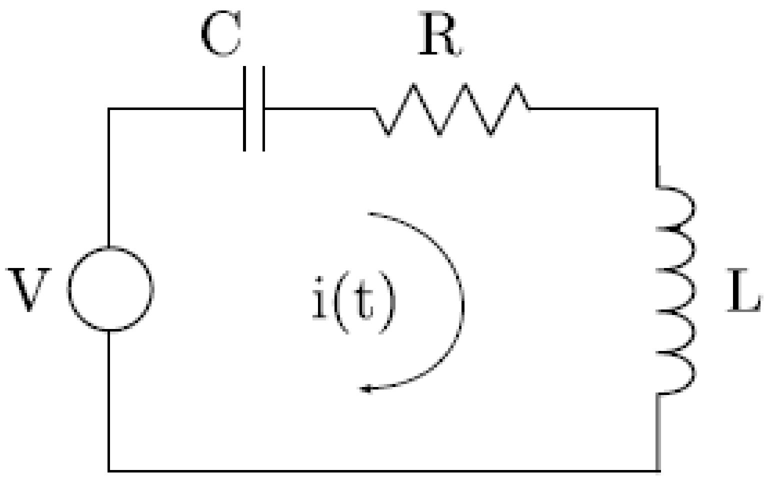


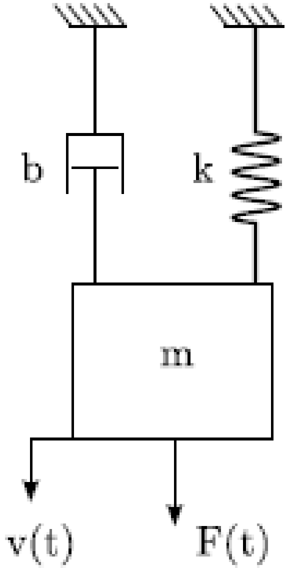


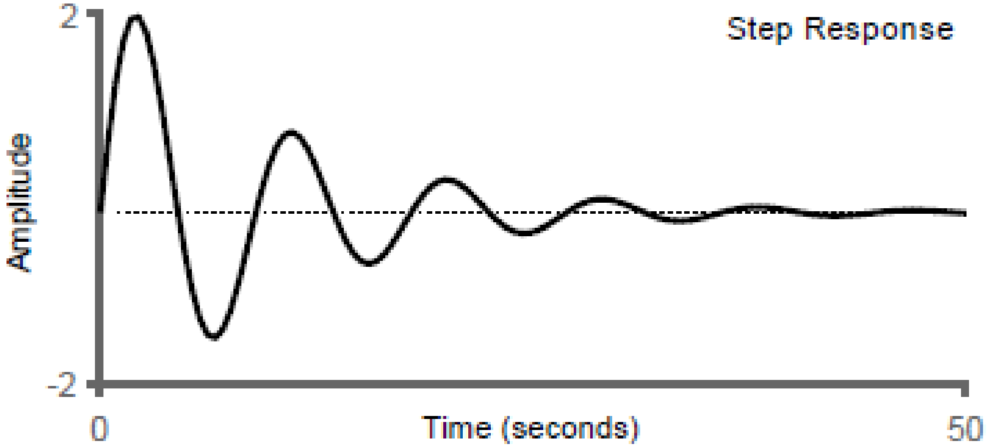


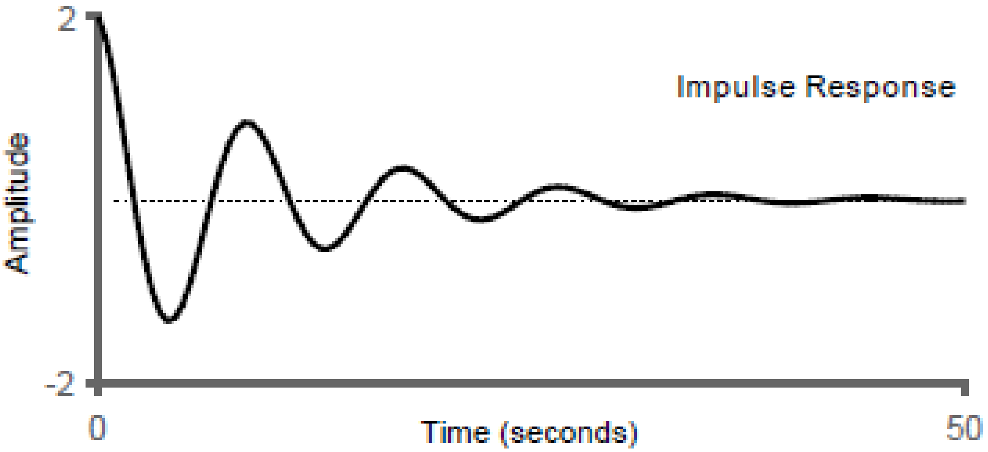
Mechanical/electrical analogy

Taking: $m = 5$ kg, $k = 0.25$ N/m, $b = 0.1$ Ns.

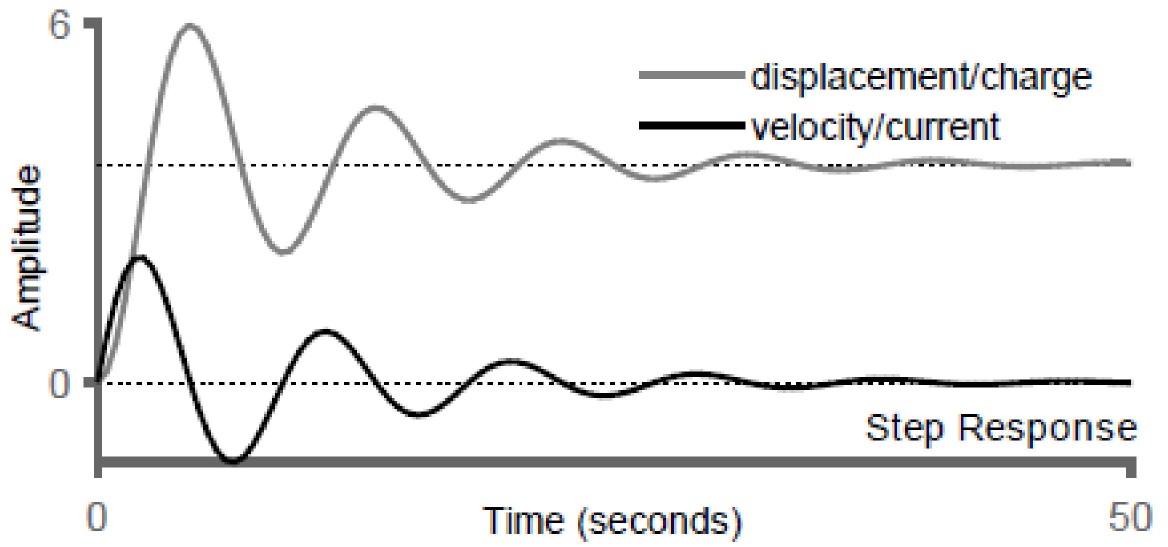








Mechanical/electrical analogy



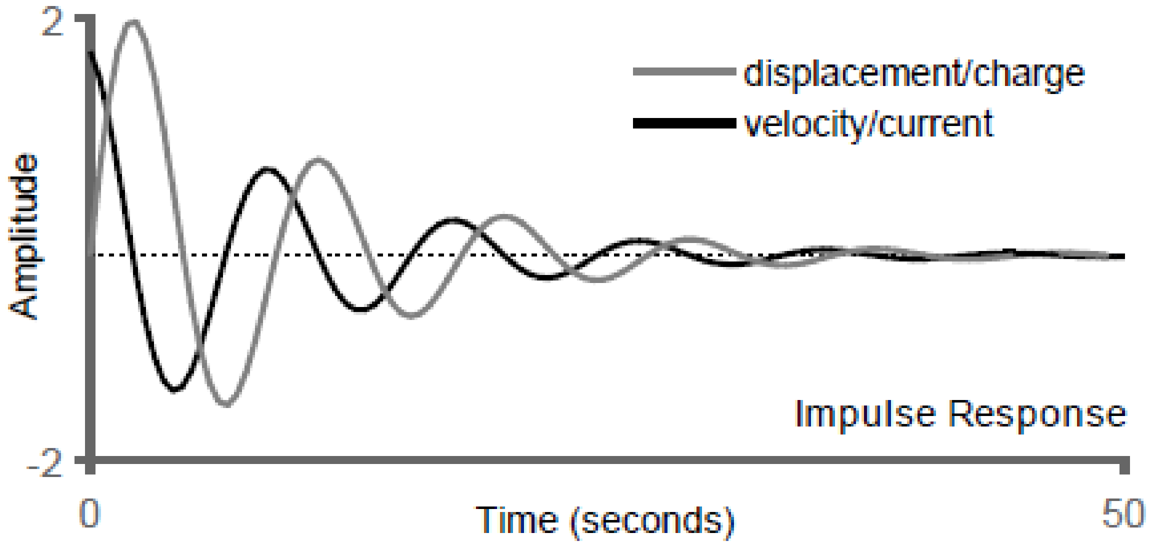
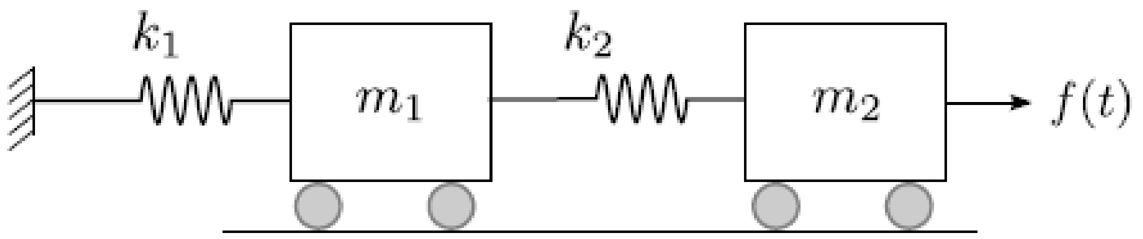


Table 2.2

summary of describing differential equation of ideal elements

Exercise 3

Find the equations of motion of the mass-spring system shown.

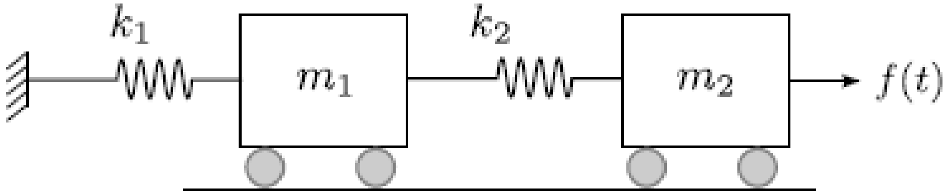


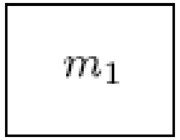
Procedure:

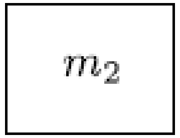
→ Draw the free body diagram of each mass

→ Apply the equation of motion

Exercise 3 - continued

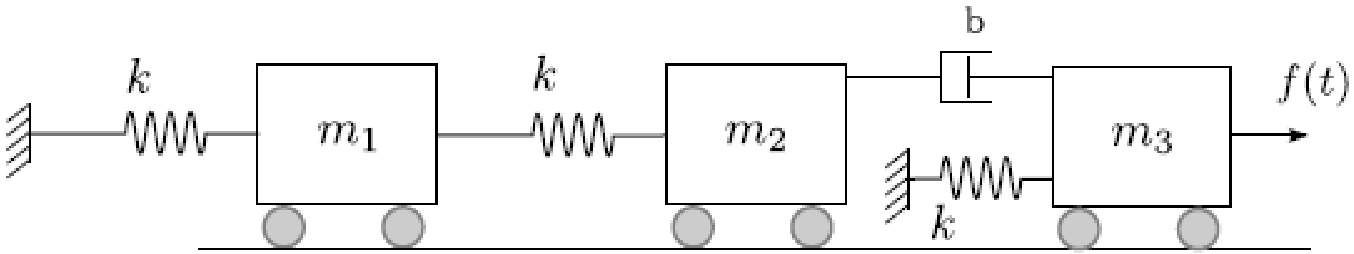






Exercise 4

Find the differential equations to model the behaviour of the system shown.

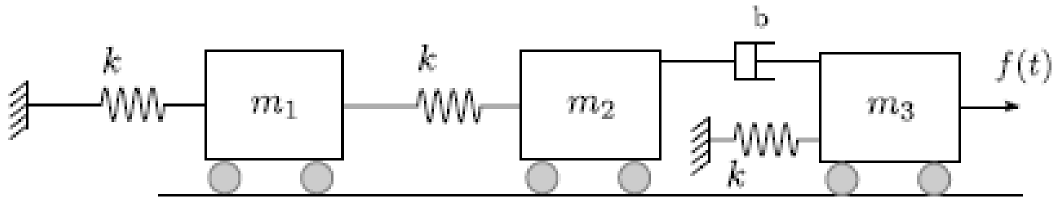


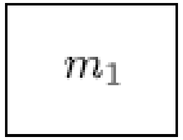
Procedure:

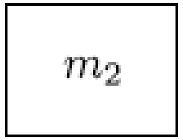
→ Draw the free body diagram of each mass

→ Apply the equation of motion

Exercise 4 - continued

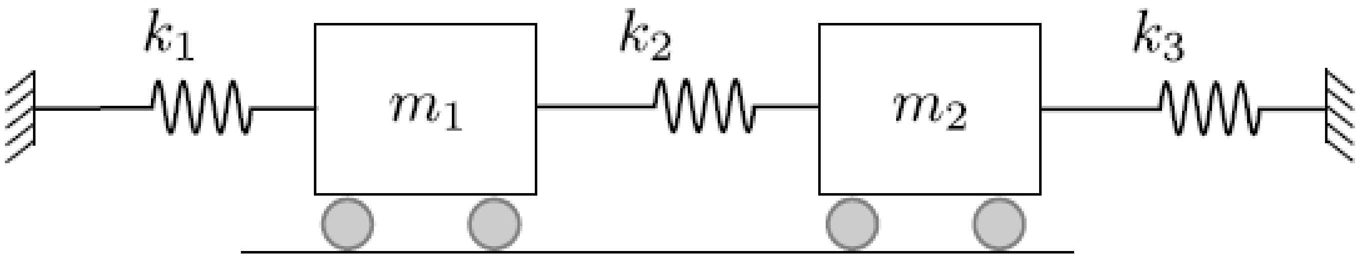






Exercise 5

Find the differential equations to model the behaviour of the system shown.

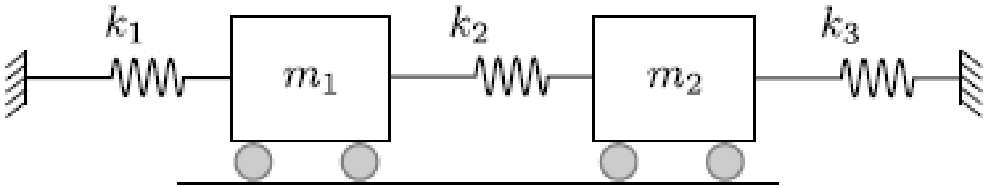


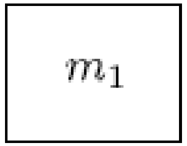
Procedure:

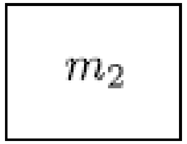
→ Draw the free body diagram of each mass

→ Apply the equation of motion

Exercise 5 - continued

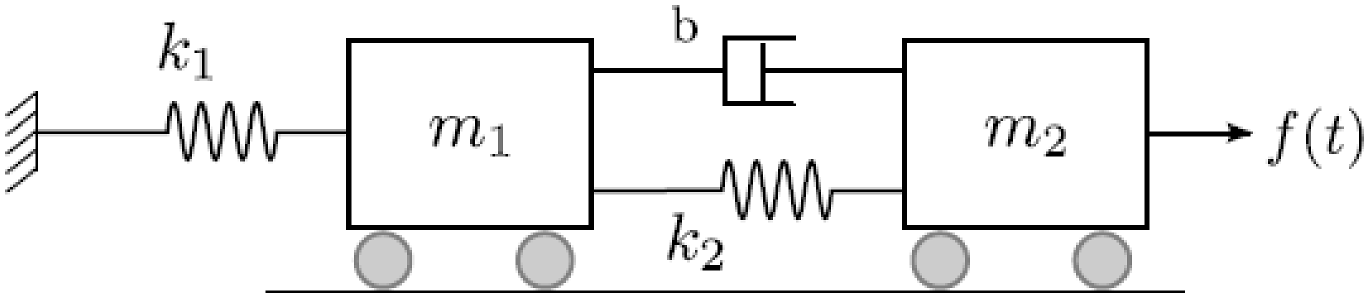






Exercise 6

Find the differential equations to model the behaviour of the system shown.

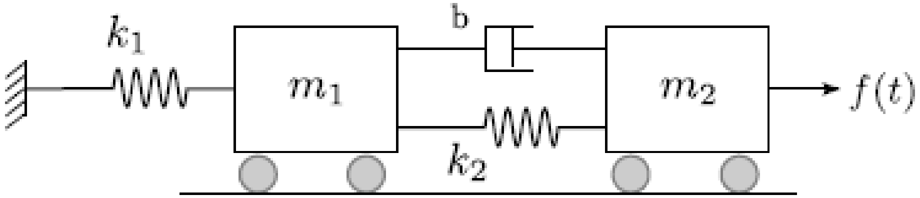


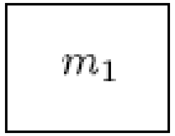
Procedure:

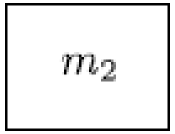
→ Draw the free body diagram of each mass

→ Apply the equation of motion

Exercise 6 - continued

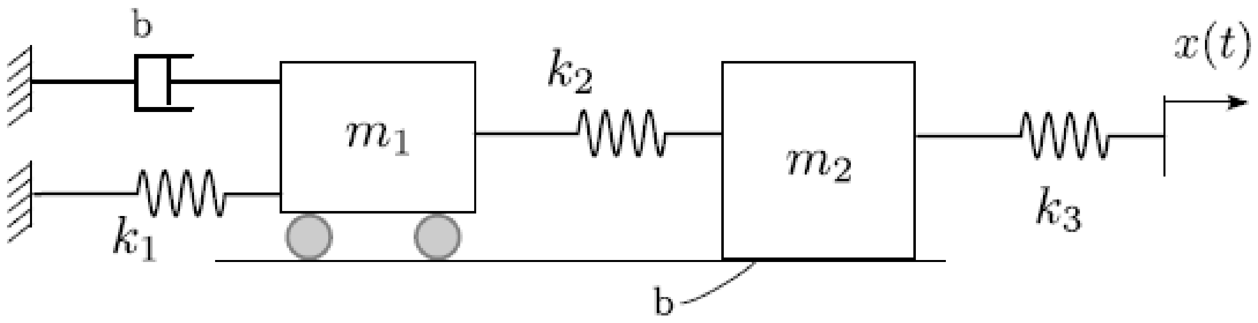






Exercise 7

Find the differential equations to model the behaviour of the system shown.

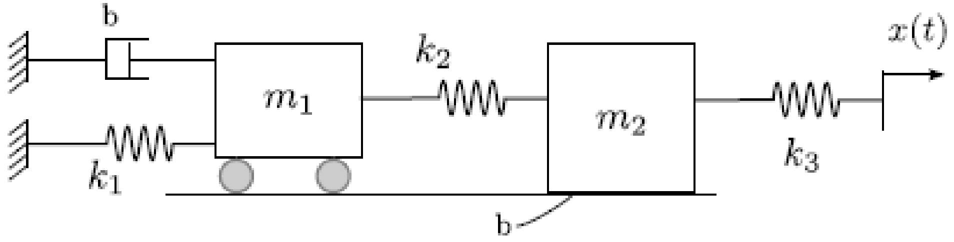


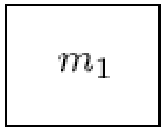
Procedure:

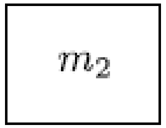
→ Draw the free body diagram of each mass

→ Apply the equation of motion

Exercise 7 - continued

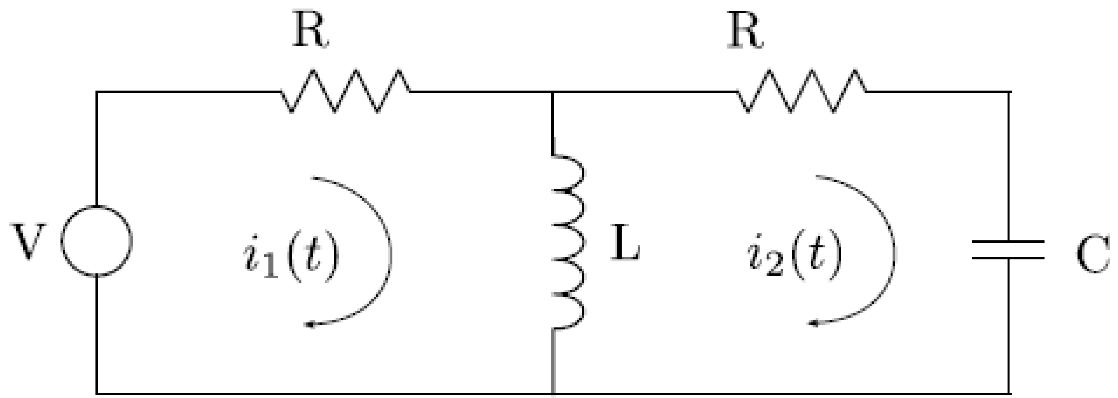






Exercise 8

Write the the differential equations ($i = f(V)$) of the following circuit.

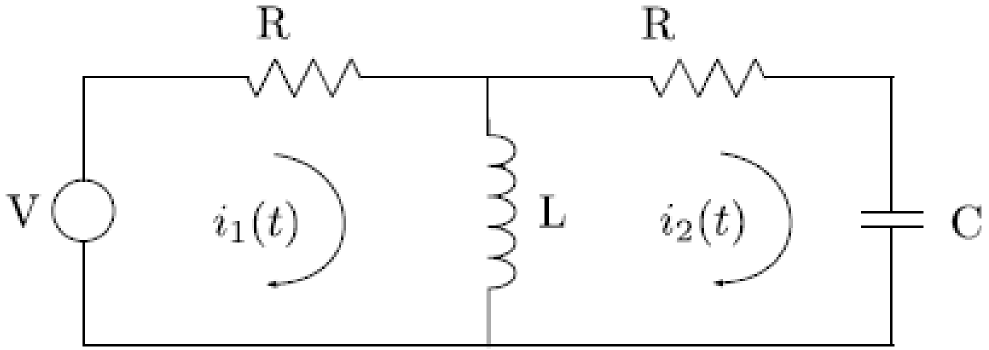


Procedure:

→ Apply Kirchhoff's voltage law

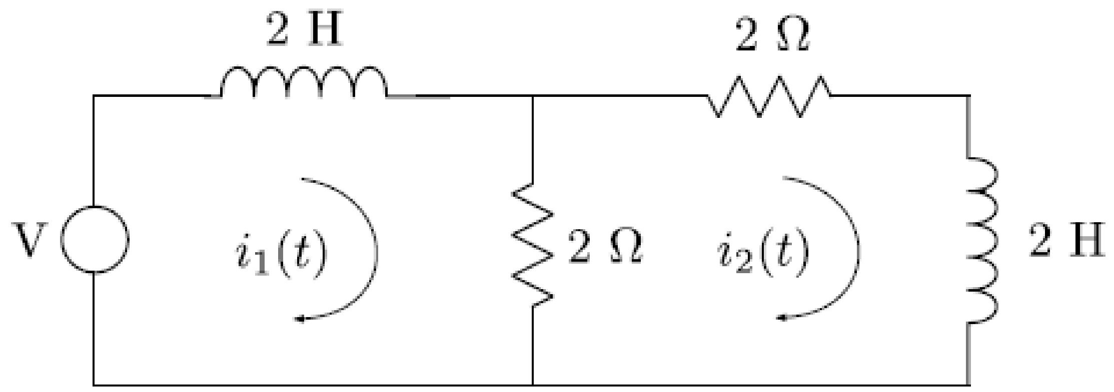
→ Find the equations for i_1 and i_2

Exercise 8 - continued



Exercise 9

Write the the differential equations ($i = f(V)$) of the following circuit.

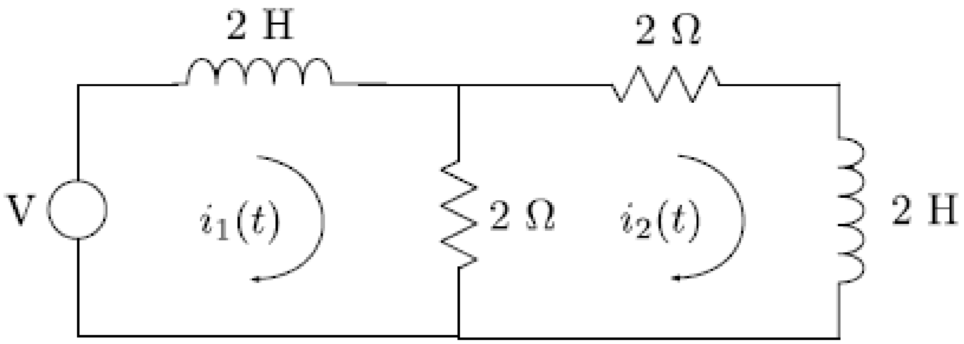


Procedure:

→ Apply Kirchhoff's voltage law

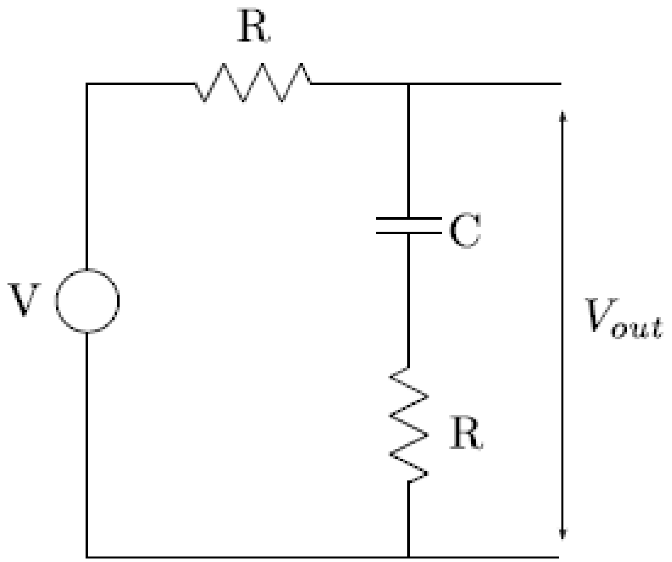
→ Find the equations for i_1 and i_2

Exercise 9 - continued



Exercise 10

Write the the differential equations ($V_{out} = f(V)$) of the following circuit.

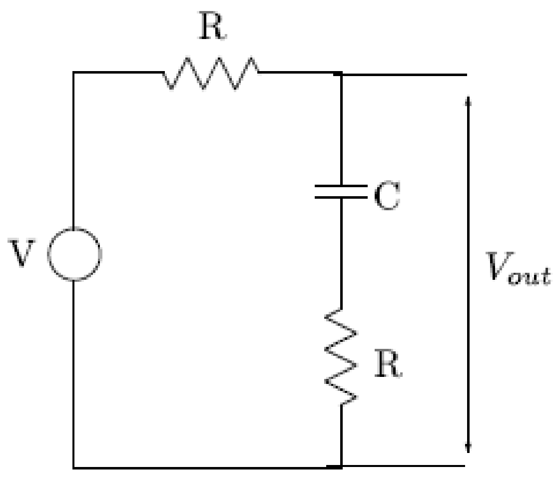


Procedure:

→ Apply Kirchhoff's law

→ Find the equations for V_{out} as a function of V

Exercise 10 - continued



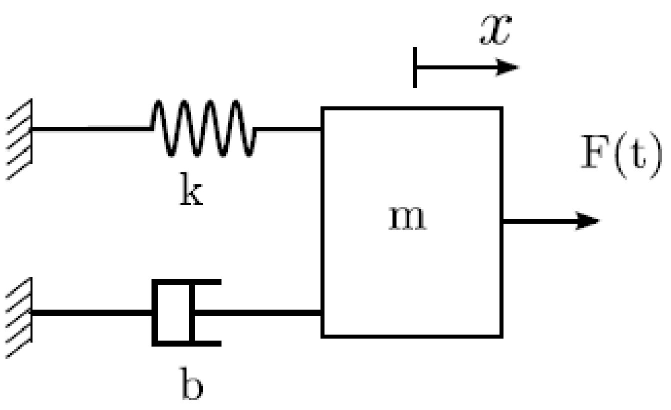
Laplace Transform

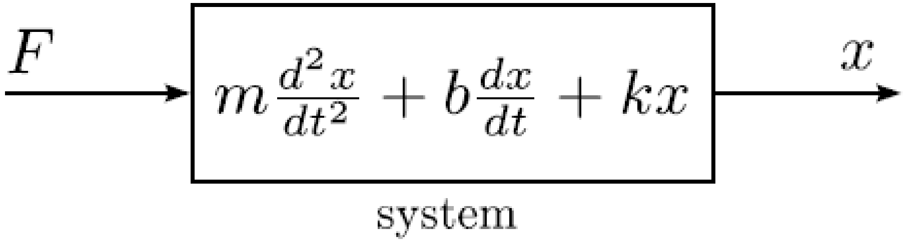
- Review the principles of the Laplace transformation
- Apply the Laplace transformation to a system

Time Domain vs. Frequency Domain

Input/output relation

Transfer function: A relation between the input and output of a given linear system





How can we evaluate the temporal response to the system?

Laplace transform

The time-response solution can be obtained using the Laplace transform.

differential equation → solution

time domain $\xrightarrow{\mathcal{L}}$ frequency domain $\xrightarrow{\mathcal{L}^{-1}}$ time domain

$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx$ $\qquad F(s) = X(s)[ms^2 + bs + k]$ $\qquad x(t) = Ke^{-\alpha t}\sin(\beta t + \theta)$

→ Obtain the linearised differential equations

→ Obtain the Laplace transformation of the differential equation

→ Solve for the variable of interest

Laplace transform

A mass-spring system is governed by the differential equation

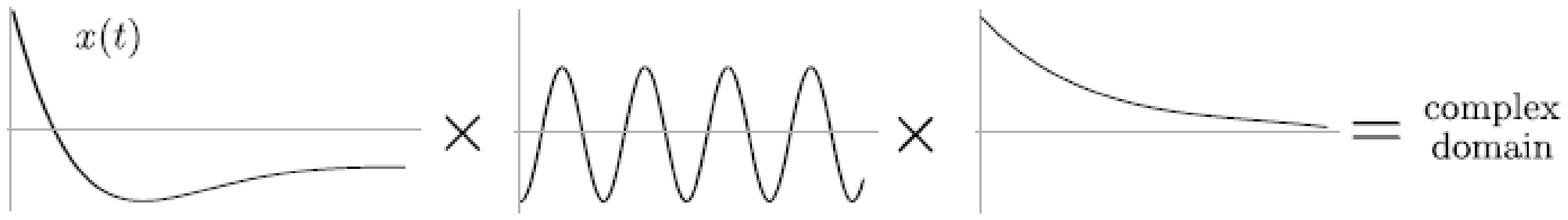
$$m\frac{d^2x(t)}{dt^2} + kx(t) = F(t) \tag{1}$$

Solutions of (1) can be

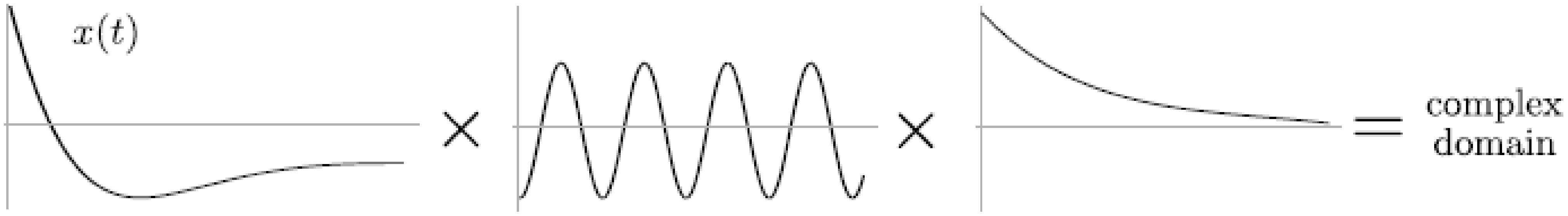
→ Exponential: $x(t) = e^{-\sigma t}$ with $\sigma \in \mathcal{R}$

→ Sinusoidal: $x(t) = \sin(\omega t) = e^{-j\omega t}$ with $j \in \mathcal{C}$

→ Exponential and sinusoidal: $x(t) = e^{-j\omega t}e^{-\sigma t}$



$$X(\sigma,\omega) = \int_{-\infty}^{\infty} \left[x(t) \cdot e^{-\sigma t} \cdot e^{-j\omega t} \right] dt$$



The standard form of the Laplace transform is:

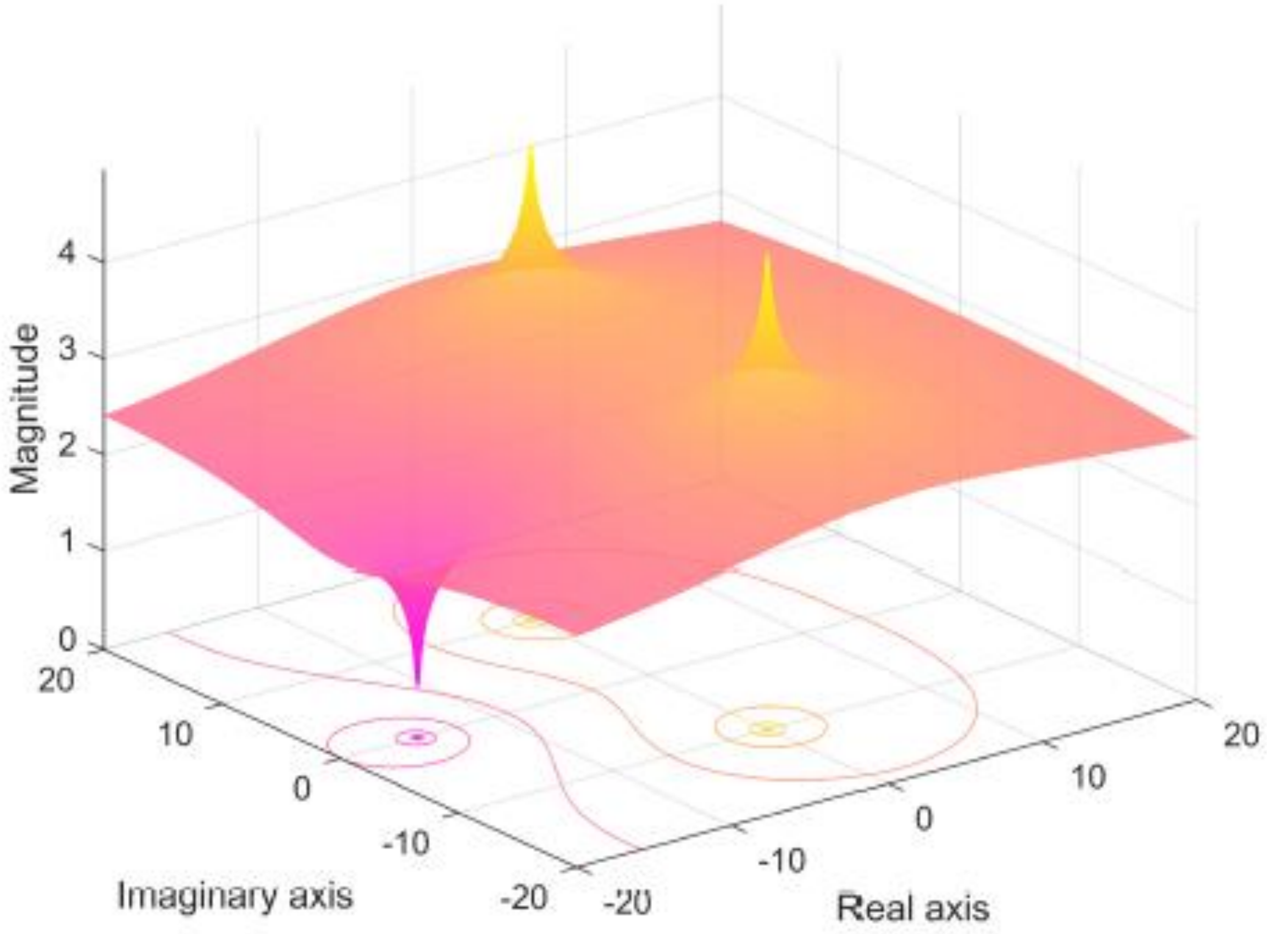
$$X(\sigma,\omega) = \int_{-\infty}^{\infty} \left[x(t) \cdot e^{(-\sigma - j\omega)t} \right] dt \quad (2)$$

where the complex variable $s = \sigma + j\omega$ has:

→ A real portion σ, which corresponds to the exponential response

→ An imaginary portion ω, which corresponds to the sinusoidal response

What is the Laplace transform?



The Laplace transformation for a function of time $f(t)$ is

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \mathscr{L}\{f(t)\} \tag{4}$$

where $s = \sigma + j\omega$.

The inverse Laplace transform is

$$f(t) = \frac{1}{2\pi j}\int_{\tau - j\infty}^{\tau + j\infty} F(s)e^{st}ds \tag{5}$$

Transfer function: The ratio of the Laplace transform of the output variable of to the input variable.

Laplace transformation

$f(t)$	$F(s)$
Impulse function $\delta(t)$	1
Step function $u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table 2.3 Important Laplace Transform Pairs

$f(t)$	$F(s)$
Step function, $u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$f^{(k)}(t) = \frac{d^k f(t)}{dt^k}$	$s^k F(s) - s^{k-1} f(0^-) - s^{k-2} f'(0^-) - \dots - f^{(k-1)}(0^-)$
$\int_{-\infty}^{t} f(t) dt$	$\frac{F(s)}{s} + \frac{1}{s} \int_{-\infty}^{0} f(t)\, dt$
Impulse function $\delta(t)$	1
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$
$\frac{1}{\omega}[(\alpha-a)^2+\omega^2]^{1/2} e^{-at} \sin(\omega t+\phi)$, $\phi = \tan^{-1} \frac{\omega}{\alpha - a}$	$\frac{s+\alpha}{(s+a)^2+\omega^2}$
$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t, \zeta < 1$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
$\frac{1}{a^2+\omega^2} + \frac{1}{\omega\sqrt{a^2+\omega^2}} e^{-at} \sin(\omega t - \phi)$, $\phi = \tan^{-1} \frac{\omega}{-a}$	$\frac{1}{s[(s+a)^2+\omega^2]}$
$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \phi)$, $\phi = \cos^{-1} \zeta, \zeta < 1$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
$\frac{\alpha}{a^2+\omega^2} + \frac{1}{\omega}\left[\frac{(\alpha-a)^2+\omega^2}{a^2+\omega^2}\right]^{1/2} e^{-at} \sin(\omega t+\phi)$, $\phi = \tan^{-1} \frac{\omega}{\alpha - a} - \tan^{-1} \frac{\omega}{-a}$	$\frac{(s+\alpha)}{s[(s+a)^2+\omega^2]}$

To illustrate the usefulness of the Laplace transform

Laplace transformation

$f(t)$	$F(s)$
$\frac{d^k f(t)}{dt^k}$	$s^k F(s) - s^{k-1} f(0) - s^{k-2} \dot{f}(0) - ... - f^{k-1}(0)$
$\int f(t) dt$	$\frac{F(s)}{s} + \frac{1}{s} \int f(t) dt \Big|_{t=0}$

The Laplace variable s can be considered to be the differential operator:

$$s \Rightarrow \frac{d}{dt} \tag{6}$$

And the integral operator:

$$\frac{1}{s} \Rightarrow \int_0^t dt \tag{7}$$

What is the Laplace transform of $f(t) = 1$?

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \mathscr{L}\{f(t)\}$$

What is the Laplace transform of $\cos(5t)$?

Laplace transform properties

Linearity

$$\mathscr{L}\{\alpha x(t)\} = \alpha \mathscr{L}\{x(t)\} = \alpha X(s) \tag{8}$$

$$\mathscr{L}\{\alpha x(t) + \beta y(t)\} = \alpha X(s) + \beta Y(s) \tag{9}$$

Time shift

$$x(t-\tau) = X(s)e^{-s\tau} \tag{10}$$

Initial value theorem

$$\lim_{t\to 0} x(t) = \lim_{s\to\infty} sX(s) \tag{11}$$

Final value theorem

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{12}$$

To illustrate the useful of Laplace Transform Let's us consider mass-spring-damper system

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = r(t)$$

Obtain response y(t) as function of time, by taking Laplace Transform using Table 2.3

When initial conditions are zero

Solving for Y(s)

$$Y(s) = \frac{(Ms + b)y_0}{Ms^2 + bs + k}$$

$$Y(s) = \frac{(Ms+b)y_0}{Ms^2+bs+k} = \frac{p(s)}{q(s)} = \frac{Numerator}{denominator} = \frac{NUM}{DEN}$$

$$Ms^2 + bs + k = 0$$

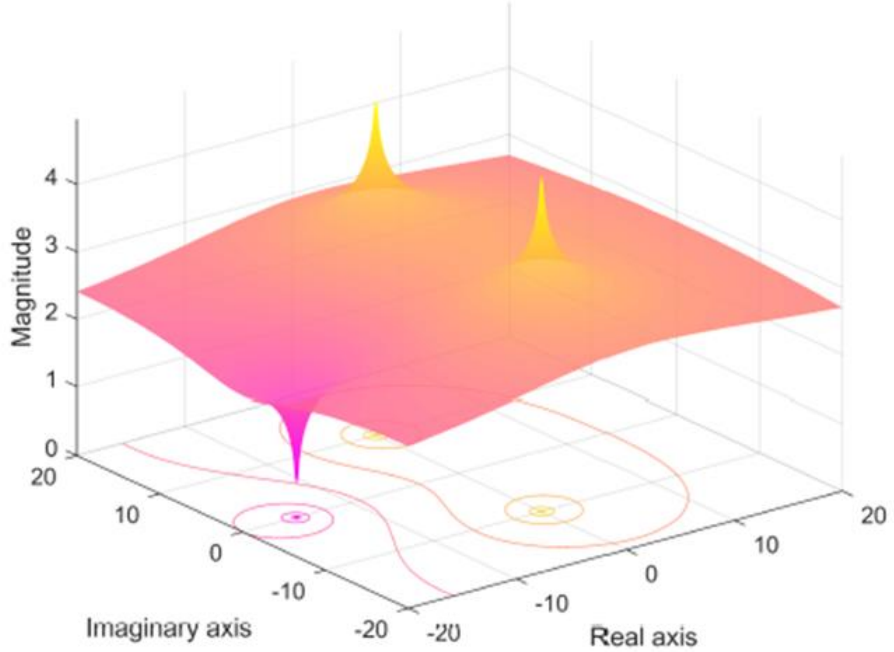
Called “Characteristic equation” since it determines the characteristic of the time response of the system

Roots are called “Poles”

$$(Ms + b)y_0$$

Roots are called “Zeros”

Y(s



S-plan Plot (2D graphic)

Let assum k/M = 2 and b/M = 3 and initial condition y_0=1 , then

$$Y(s) = \frac{(s+3)(1)}{(s+1)(s+2)}$$

Partial fraction

$$Y(s) = \frac{k1}{(s+1)} + \frac{k2}{(S+2)}$$

$$Y(s) = \frac{2}{(s+1)} + \frac{-1}{(S+2)}$$

Taking Inverse Laplace
Using table 2.3

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-1}{(s+2)}\right\}$$

Using table 2.3

$$y(t) = 2e^{-t} - 1e^{-2t}$$

Finally, as time,t goes to infinity. *Y(t) = 0*

Hence, the final position of mass in equilibrium is y = 0

Final Value Theorem

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

In our case,

$$Y(s) = \frac{(s+3)(1)}{(s+1)(s+2)} = 0$$

Hence, the final position of mass in equilibrium is y = 0

Common forcing signals

Impulse function

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases} \rightarrow I(s) = A$$

Step function

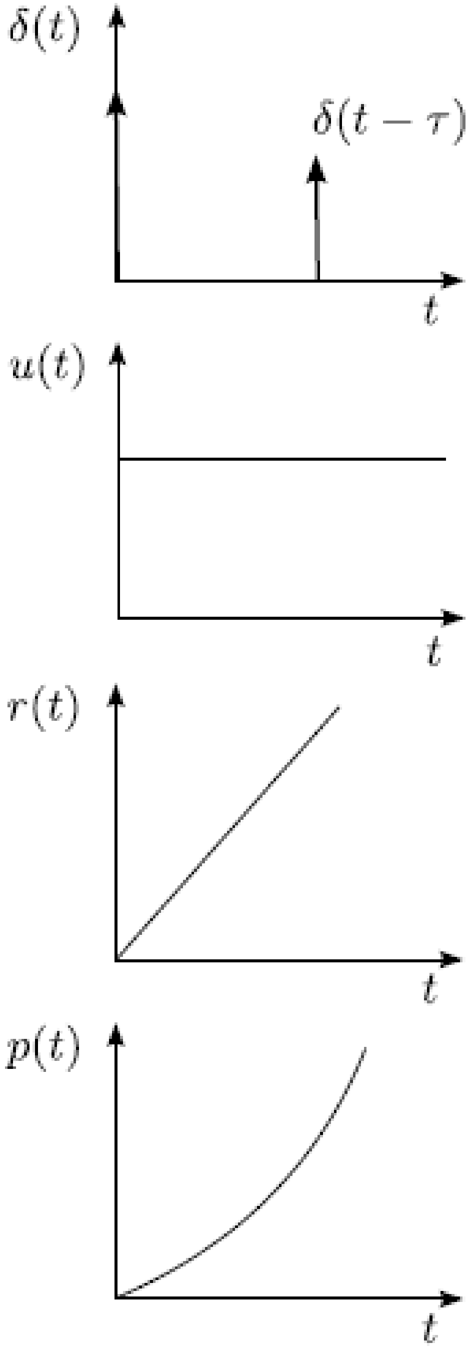
$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow U(s) = A\frac{1}{s}$$

Ramp function

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow R(s) = A\frac{1}{s^2}$$

Parabolic function

$$p(t) = \begin{cases} A\frac{t^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow P(s) = A\frac{1}{s^3}$$



Partial fraction decomposition

Consider the function $F(s)$ given by

$$F(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{n} a_k s^k}{\sum_{p=0}^{p} a_p s^p} \tag{15}$$

with A and B being polynomials **and** $\mathbf{p} > \mathbf{k}$.

How to split up a complicated fraction into known forms such as:

$$F(s) = \frac{c_1}{s+a_1} + \frac{c_2}{s+a_2} \dots \frac{a_p}{s+c_p} \ ?$$

Such that:

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{c_1}{s+a_1} + \frac{c_2}{s+a_2} \dots + \frac{c_p}{s+a_p}\right\}$$

$$f(t) = \mathscr{L}^{-1}\left\{\frac{c_1}{s+a_1}\right\} + \mathscr{L}^{-1}\left\{\frac{c_2}{s+a_2}\right\} \dots + \mathscr{L}^{-1}\left\{\frac{c_p}{s+a_p}\right\}$$

Partial fraction decomposition

For each factor in the denominator, the term in the decomposition is:

Factor in denominator	Term in partial decomposition
$as + b$	$\frac{c}{as+b}$
$(as + b)^k$	$\frac{c_1}{as+b} + \frac{c_2}{(as+b)^2} + \cdots \frac{c_k}{(as+b)^k}$
$as^2 + bx + d$	$\frac{c_1 s+c_2}{as^2+sb+d}$
$(as^2 + bx + d)^k$	$\frac{c_1 s+e_1}{as^2+sb+d} + \frac{c_2 s+e_2}{(as^2+sb+d)^2} + \cdots \frac{c_k s+e_k}{(as^2+sb+d)^k}$

Partial fraction decomposition

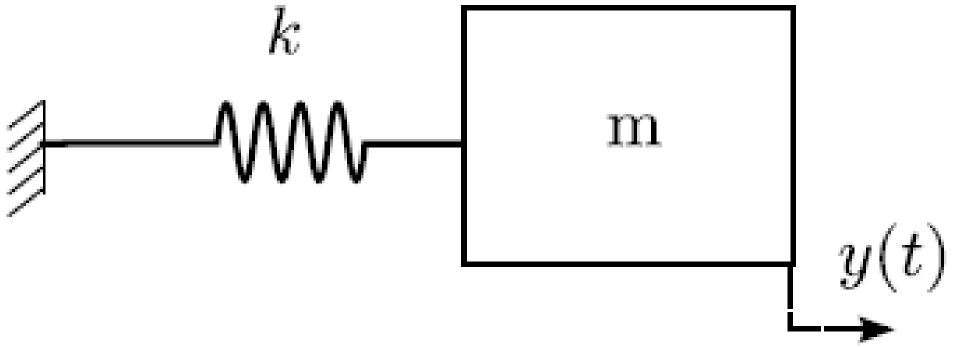
$$\frac{s^2+15}{(s+3)^2(s^2+3)}$$

Laplace transformation

$F(s)$	$f(t)$
1	Unit impulse $\delta(t)$
$\frac{1}{s}$	Unit step function $u(t)$
$\frac{1}{s^2}$	Unit ramp t
$\frac{n!}{s^{n+1}}$	t^n with $n \in \mathbb{N}^+$
$\frac{1}{s+a}$	e^{-at}
$\frac{1}{(s+a)^2}$	te^{-at}
$\frac{s}{(s+a)(s+b)}$	$\frac{1}{b-a}\left(be^{-bt} - ae^{-at}\right)$

Exercise 11

If the mass is released from rest when the spring is stretched by $y(0) = \alpha$, calculate its position $y(t)$ as a function of time.



Procedure:

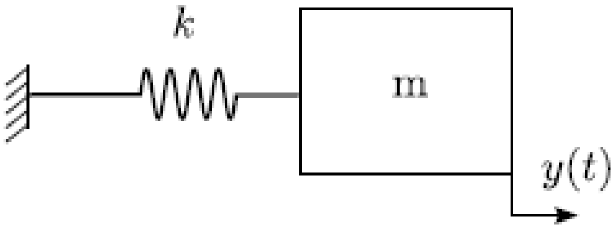
→ Find the differential equation

→ Calculate the Laplace transform

→ Determine the temporal response using the inverse transformation

Exercise 11 - continued

Given $y(0) = \alpha$, $\dot{y}(0) = 0$. Determine $y(t)$.



Exercise 12

Consider the following differential equation in the frequency domain

$$F(s) = \frac{10}{s(s+1)(s+10)}$$

Determine:

$\rightarrow$ The final value of the function $f(t)$ when $t \rightarrow \infty$

$\rightarrow$ The function $f(t)$

Exercise 12 - continued

Final value

$$F(s) = \frac{10}{s(s+1)(s+10)}$$

$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$

Exercise 12 - continued

Partial fraction expansion

$$F(s) = \frac{10}{s(s+1)(s+10)} =$$

Exercise 12 - continued

Inverse transformation

$$F(s) = \frac{1}{s} - \frac{10}{9}\frac{1}{s+1} + \frac{1}{9}\frac{1}{s+10}$$

Exercise 13

Consider the following differential equation in the frequency domain

$$F(s) = \frac{1}{s(s+2)^2}$$

Determine:

→ The final value of the function $f(t)$ when $t \to \infty$

→ The function $f(t)$

Exercise 13 - continued

Final value

$$F(s) = \frac{1}{s(s+2)^2}$$

$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$

Exercise 13 - continued

Inverse transformation

$$F(s) = \frac{1}{4s} - \frac{1}{4}\frac{1}{s+2} - \frac{1}{2}\frac{1}{(s+2)^2}$$

Exercise 14

Assuming all initial conditions are zero, determine the solution of the following differential equation, where the forcing term is $f(t) = 2e^{-t}$.

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y(t) = f(t)$$

Exercise 14 - continued

Exercise 14 - continued

$$Y(s) = \frac{2}{(s+1)^3} - \frac{2}{(s+1)^2} + \frac{2}{s+1} - \frac{2}{s+2}$$

Exercise 15

A laser printer uses a laser beam to print copy rapidly for a computer. The laser is positioned by a control input $r(t)$ so that we have:

$$Y(s) = \frac{6(s+50)}{s^2+40s+300}R(s).$$

The input $r(t)$ represents the desired position of the laser beam. If $r(t) = 1$, determine:

$\rightarrow$ The output $y(t)$

$\rightarrow$ The final value of $y(t)$

Exercise 15 - continued

1 - Partial fraction expansion

$$R(s) = \frac{1}{s}, \quad Y(s) = \frac{6(s+50)}{s^2+40s+300} R(s).$$

2 - Inverse transform

Exercise 15 - continued

2 - Final value of $y(t)$

$$R(s) = \frac{1}{s}, \quad Y(s) = \frac{6(s+50)}{s^2+40s+300}R(s).$$

Next lecture

- Transfer Function
- Block Diagram
- Computer Simulation